

ALIASING FREE FOR MIXED SPECTRA FOR STABLE PROCESSES

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ABSTRACT

This work focuses on the symmetric alpha stable processes with continuous time frequently used in modeling the signal with indefinitely growing variance when the spectral measure is mixed: sum of a continuous measure and discrete measure. The objective of this paper is to estimate the spectral density of the continuous part from discrete observations of the signal. For that, we propose a method based on a sample of the signal at a periodic instant. The Jackson polynomial kernel is used for construct a periodogram. We smooth this periodogram by two spectral windows taking into account the width of the interval where the spectral density is non-zero. This technique allows to circumvent the phenomenon of aliasing often encountered in the estimation from the discrete observations of a process with a continuous time.

KEYWORDS

Spectral density, stable processes, periodogram, smoothing estimate, aliasing

1. INTRODUCTION

Stable alpha processes have been of interest to several research authors for their multiple applications when we have random signals with variance indefinitely increase. The harmonizable process is an important example of a symmetric α -stable process, and its proprieties have been considered by numerous authors like [1]-[10] to name a few.

In particular, stable symmetric processes find their place in various applications and in various fields such as: physics, biology, electronics and electricity, hydrology, economies, communications and radar applications, ...ect. See: [11]-[22]. This work considers a symmetric alpha stable harmonizable process $X = \{X(t) : t \in \mathbb{R}\}$. Alternatively X has the integral representation:

$$X(t) = \int \exp[i(t\lambda)] d\xi(\lambda) \quad (1)$$

where $1 < \alpha < 2$ and ξ is a complex valued symmetric α -stable random measure on \mathbb{R} with independent and isotropic increments. The measure defined by $m(A) = |\xi(A)|_\alpha^\alpha$ (see [4]) is called "control" measure or spectral measure." The spectral density function was already estimated in different cases: by E.Masry and S.Combanis [4] when the time of the process is continuous, by Sabre [23] when the time of the process is discrete and by R. Sabre [24]-[25] when the time of the process is p-adic.

This work considers a general case where we suppose that the spectral measure is the sum of an absolutely continuous measure with respect to Lebesgue measure and a discrete measure:

$$d\mu(\lambda) = \phi(x)dx + \sum_{i=1}^q c_i \delta_{w_i}$$

where δ is a Dirac measure, ϕ is nonnegative integrable and bounded function. c_i is an unknown positive real number and w_i is an unknown real number. Assume that $w_i \neq 0$. The function ϕ is called the spectral density. Discrete measure is due to random repeated value jumps during experimental measurements. The spectral density ϕ represents the distribution of the energy carried by the signal.

Our goal is to establish a non-parametric estimate of the spectral density ϕ from discrete observations of $X(t)$. This is motivated by the fact that, in practice, it is not obvious to observe the process on continuous interval of time. Indeed, we sampled the process at instants t_n , equally distant, i.e., $t_n = n\tau$, $\tau > 0$. It is known that aliasing of ϕ occurs. For more details about aliasing phenomenon, see [26]. To avoid this difficulty, we suppose that the spectral density ϕ is vanishing for $|\lambda| > \Omega$ where Ω is a nonnegative real number. From some smoothing, we construct an estimate depending on Ω and we show that it is asymptotically unbiased and consistent.

Briefly, we indicate the organisation of this paper: the outline in this paper is as follows: we present in the second section two technical lemmas, the preiodogram and show that this periodogram is asymptotically unbiased estimated but not consistent. In the third section, we smooth this periodogram by two chosen spectral windows to estimate the spectral density at jump points. We show that the smoothing periodogram is a consistent estimator.

2. THE PERIODOGRAM AND ITS PROPRIETIES

First, we introduce some basic notations and properties of the Jackson's polynomial kernel. Let N is the size of sample of X . Let k and n are the numbers satisfying:

$$N-1 = 2k(n-1) \quad \text{with} \quad n \in N \quad k \in N \cup \left\{ \frac{1}{2} \right\} \quad \text{if} \quad k = \frac{1}{2} \quad \text{then} \quad n = 2n_1 - 1, \quad n_1 \in N.$$

The Jackson's polynomial kernel is defined by: $|H_N(\lambda)|^\alpha = |A_N H^{(N)}(\lambda)|^\alpha$ where

$$H^{(N)}(\lambda) = \frac{1}{q_{k,n}} \left(\frac{\sin \frac{n\lambda}{2}}{\sin \frac{\lambda}{2}} \right)^{2k} \quad \text{with} \quad q_{k,n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{\sin \frac{n\lambda}{2}}{\sin \frac{\lambda}{2}} \right)^{2k} d\lambda.$$

where $A_N = (B_{\alpha,N})^{-\frac{1}{\alpha}}$ with $B_{\alpha,N} = \int_{-\pi}^{\pi} |H^{(N)}(\lambda)|^{\alpha} d\lambda$.

We give the following lemmas which are used in the rest of this paper. Their proof are given in [23].

Lemma 2.1 *There is a non negative function h_k such as:*

$$H^{(N)}(\lambda) = \sum_{m=-k(n-1)}^{k(n-1)} h_k\left(\frac{m}{n}\right) \cos(m\lambda)$$

$$B'_{\alpha,N} = \int_{-\pi}^{\pi} \left| \frac{\sin \frac{n\lambda}{2}}{\sin \frac{\lambda}{2}} \right|^{2k\alpha} d\lambda \quad \text{and} \quad J_{N,\alpha} = \int_{-\pi}^{\pi} |u|^{\gamma} |H_N(u)|^{\alpha} du,$$

Lemma 2.2 *Let*

where

$\gamma \in]0, 2]$, *then*

$$B'_{\alpha,N} \begin{cases} \geq 2\pi \left(\frac{2}{\pi}\right)^{2k\alpha} n^{2k\alpha-1} & \text{if } 0 < \alpha < 2 \\ \leq \frac{4\pi k\alpha}{2k\alpha-1} n^{2k\alpha-1} & \text{if } \frac{1}{2k} < \alpha < 2 \end{cases}$$

And

$$J_{N,\alpha} \leq \begin{cases} \frac{\pi^{\gamma+2k\alpha}}{2^{2k\alpha}(\gamma-2k\alpha+1)} \frac{1}{n^{2k\alpha-1}} & \text{if } \frac{1}{2k} < \alpha < \frac{\gamma+1}{2k} \\ \frac{2k\alpha\pi^{\gamma+2k\alpha}}{2^{2k\alpha}(\gamma+1)(2k\alpha-\gamma-1)} \frac{1}{n^{\gamma}} & \text{if } \frac{\gamma+1}{2k} < \alpha < 2 \end{cases}$$

In this section, we give a periodogram and we develop its proprieties. Assume that the process

$X(t)$, defined in (1), is observed at instants $t_j = j\tau$, $j = 1, 2, \dots, N$ and $\tau = \frac{2\pi}{\omega}$, where ω is a real number strictly greater than 2Ω . We define the periodogram \hat{I}_N on $]-\Omega, \Omega[$ as follows:

$$\hat{I}_N(\lambda) = C_{p,\alpha} |I_N(\lambda)|^p, \quad 0 < p < \frac{\alpha}{2}$$

where

$$I_N(\lambda) = [\tau]^{\frac{1}{\alpha}} A_N \text{Re} \left[\sum_{n'=-k(n-1)}^{n'=k(n-1)} h_k\left(\frac{n'}{n}\right) \exp\{-i(n'\tau\lambda)\} X(n'\tau + k(n-1)\tau) \right],$$

and the normalisation constant $C_{p,\alpha}$ is given by $C_{p,\alpha} = \frac{D_p}{F_{p,\alpha}[C_\alpha]^{p/\alpha}}$, with

$$D_p = \int_{-\infty}^{\infty} \frac{1 - \cos(u)}{|u|^{1+p}} du \quad \text{and} \quad F_{p,\alpha} = \int_{-\infty}^{\infty} \frac{1 - e^{-|u|^\alpha}}{|u|^{1+p}} du$$

Lemma 2.3

The characteristic function of $I_N(\lambda)$, $E \exp[irI_N(\lambda)]$, converges to $\exp[-C_\alpha |r|^\alpha (\psi_{N,1}(\lambda) + \psi_{N,2}(\lambda))]$, where

$$\psi_{N,1}(\lambda) = \int_{-\pi}^{\pi} |H_N(y - \tau\lambda)|^\alpha \phi\left(\frac{y}{\tau}\right) dy \quad \text{and} \quad \psi_{N,2}(\lambda) = \sum_{i=1}^q c_i |H_N(w_i - \tau\lambda)|^\alpha$$

Proof

By substituting (1) in the expression of I_N , we have:

$$I_N(\lambda) = [\tau]^\alpha A_N \text{Re} \int_{R'} \sum_{n=-k(n-1)}^{n=k(n-1)} h_k \left(\frac{n}{n} \right) \exp\{i[n'\tau(\lambda - u)]\} \exp\{i[\tau uk(n-1)]\} d\xi(u).$$

It follows from [1] and the definition of the Jackson polynomial kernel that the characteristic function is the form:

$$E \exp[irI_N(\lambda)] = \exp[-C_\alpha |r|^\alpha \psi_N(\lambda)]. \tag{2}$$

where $\psi_N(\lambda) = \psi_{N,1}(\lambda) + \psi_{N,2}(\lambda)$ with

$$\psi_{N,1}(\lambda) = \int_R |H_N(v - \tau\lambda)|^\alpha \phi\left(\frac{v}{\tau}\right) dv \quad \text{and} \quad \psi_{N,2}(\lambda) = \sum_{i=1}^q c_i |H_N(w_i - \tau\lambda)|^\alpha$$

We can write
$$\psi_{N,1}(\lambda) = \int_R |H_N(v - \tau\lambda)|^\alpha \phi\left(\frac{v}{\tau}\right) dv = \sum_{j \in Z} \int_{(2j-1)\pi}^{(2j+1)\pi} |H_N(v - \tau\lambda)|^\alpha \phi\left(\frac{v}{\tau}\right) dv.$$

Putting $v = y - 2\pi j$ and using the fact that H_N is 2π -periodic, we obtain

$$\psi_N(\lambda) = \sum_{j \in Z} \int_{-\pi}^{\pi} |H_N(y - \tau\lambda)|^\alpha \phi_j(y) dy, \quad \text{where} \quad \phi_j(y) = \phi\left(\frac{y}{\tau} - \frac{2\pi}{\tau} j\right). \quad \text{Let } j \text{ be an integer}$$

such that $-\Omega < \frac{y-2\pi j}{\tau} < \Omega$. Using the fact that $\tau\Omega < \pi$ and $|y| < \pi$, we get $|j| < \frac{\tau\Omega}{2\pi} + \frac{1}{2} < 1$ and then $j = 0$. Consequently:

$$\psi_{N,1}(\lambda) = \int_{-\pi}^{\pi} |H_N(y - \tau\lambda)|^\alpha \phi\left(\frac{y}{\tau}\right) dy. \tag{3}$$

Theorem 2.4 Let $-\Omega < \lambda < \Omega$ then $E[\hat{I}_N(\lambda)] = [\psi_N(\lambda)]^{\frac{p}{\alpha}}$,

Proof

As in [4], we use the following equality: for all real x and $0 < p < 2$,

$$|x|^p = D_p^{-1} \int_{-\infty}^{\infty} \frac{1 - \cos(xu)}{|u|^{1+p}} du = D_p^{-1} \operatorname{Re} \int_{-\infty}^{\infty} \frac{1 - e^{ixu}}{|u|^{1+p}} du, \tag{4}$$

replacing x by I_N , we obtain

$$\hat{I}_N(\lambda) = \frac{1}{F_{p,\alpha}[C_a]^{p/\alpha}} \operatorname{Re} \int_{-\infty}^{\infty} \frac{1 - \exp\{iuI_N(\lambda)\}}{|u|^{1+p}} du, \tag{5}$$

Using (6) and the definition of the $F_{p,\alpha}$, we get

$$\begin{aligned} E\hat{I}_N(\lambda) &= \frac{1}{F_{p,\alpha}[C_a]^{p/\alpha}} \int_R \frac{1 - \exp\{-C_a |u|^\alpha \psi_N(\lambda)\}}{|u|^{1+p}} du. \\ &= [\psi_N(\lambda)]^{p/\alpha}. \end{aligned}$$

3. SMOOTHING PERIODOGRAM

In order to obtain a consistent estimate of $[\phi(\lambda)]^{\frac{p}{\alpha}}$, we smooth the periodogram via spectral windows depending on whether $\tau\lambda$ is a jump point or not ($\tau\lambda \neq w_i$).

$$f_N(\lambda) = \begin{cases} f_N^{(1)}(\lambda) & \text{if } \tau\lambda \notin \{w_1, w_2, \dots, w_q\} \\ \frac{f_N^{(2)}(\lambda) - cf_N^{(1)}(\lambda)}{1-c} & \text{else} \end{cases}$$

where $f_N^{(1)}(\lambda) = \int_{-\pi}^{\pi} W_N^{(1)}(\lambda - u) \hat{I}_N(u) du$ and $f_N^{(2)}(\lambda) = \int_{-\pi}^{\pi} W_N^{(2)}(\lambda - u) \hat{I}_N(u) du$.

The spectral windows $W_N^{(1)}$ and $W_N^{(2)}$ are defined by: $W_N^{(1)}(x) = M_N^{(1)}W(M_N^{(1)}x)$ and $W_N^{(2)}(x) = M_N^{(2)}W(M_N^{(2)}x)$ with W is an even nonnegative, continuous function, vanishing for

$|\lambda| > 1$ such that $\int_{-1}^1 W(u)du = 1$. The bandwidths $M_N^{(1)}$ and $M_N^{(2)}$ satisfying: $c = \frac{M_N^{(2)}}{M_N^{(1)}}$
 $\lim_{N \rightarrow \infty} M_N^{(i)} = +\infty$, $\lim_{N \rightarrow \infty} \frac{M_N^{(i)}}{N} = 0$ for $i=1,2$, $\lim_{N \rightarrow \infty} \frac{M_N^{(2)}}{M_N^{(1)}} = 0$ and such that
 $W(M_N^{(2)}\theta) = W(M_N^{(1)}\theta) \forall \theta \in \left[-\frac{1}{M_N^{(1)}}, \frac{1}{M_N^{(1)}}\right]$.

We first show that $f_N(\lambda)$ is an asymptotically unbiased estimator of $[\phi(\lambda)]^{\frac{p}{\alpha}}$ for $-\Omega < \lambda < \Omega$ and $\tau\lambda \notin \{w_1, w_2, \dots, w_q\}$.

Theorem 3.1

Let $-\Omega < \lambda < \Omega$, such that $\tau\lambda \notin \{w_1, w_2, \dots, w_q\}$. Then, $E[f_N(\lambda)] - [\phi(\lambda)]^{\frac{p}{\alpha}} = o(1)$.

If ϕ satisfies the hypothesis $|\phi(x) - \phi(y)| \leq cste|x - y|^{-\gamma}$, with $\gamma < 2k\alpha - 1$, then,

$$E[f_N(\lambda)] - [\phi(\lambda)]^{\frac{p}{\alpha}} = \begin{cases} O\left(\frac{1}{n^{(2k\alpha-1)}} + \frac{1}{M_N^{(1)\gamma}}\right) & \text{if } \lambda \neq 0 \\ O\left(\frac{1}{M_N^{(1)}n^{(2k\alpha-1)}} + \frac{1}{M_N^{(1)\gamma}} + \frac{1}{n^{2k\alpha-1}}\right) & \text{if } \lambda = 0 \end{cases}.$$

Proof

By the definition of the spectral window, we have:

$$E[f_N(\lambda)] = \int_R M_N^{(1)}W[M_N^{(1)}(\lambda - u)]E[\hat{I}_N(u)]du.$$

Let $M_N^{(1)}(\lambda - u) = v$ and from (11), we obtain:

$$E[f_N(\lambda)] = \int_{-1}^1 W(v) \left[\psi_N \left(\lambda - \frac{v}{M_N^{(1)}} \right) \right]^{\frac{p}{\alpha}} dv. \quad (7)$$

Using the fact that $\int_{-1}^1 W(u)du = 1$ and the inequality (4), we get:

$$\left| E[f_N(\lambda)] - [\phi(\lambda)]^\alpha \right| \leq \int_{-1}^1 W(v) \left| \psi_N \left(\lambda - \frac{v}{M_N^{(1)}} \right) - \phi(\lambda) \right|^\alpha dv.$$

Since $\frac{p}{\alpha} < 1$, we obtain

$$\left| \psi_N \left(\lambda - \frac{v}{M_N^{(1)}} \right) - \phi(\lambda) \right|^\alpha \leq \left| \psi_{N,1} \left(\lambda - \frac{v}{M_N^{(1)}} \right) - \phi(\lambda) \right|^\alpha + \left| \psi_{N,2} \left(\lambda - \frac{v}{M_N^{(1)}} \right) \right|^\alpha$$

We now examine the limit of $\psi_{N,1} \left(\lambda - \frac{v}{M_N^{(1)}} \right)$. From (3) we get:

$$\psi_{N,1} \left(\lambda - \frac{v}{M_N^{(1)}} \right) = \int_{-\infty}^{\infty} \left| H_N \left(u - \tau \left(\lambda - \frac{v}{M_N^{(1)}} \right) \right) \right|^\alpha \phi \left(\frac{u}{\tau} \right) du.$$

Let $u - \tau \left(\lambda - \frac{v}{M_N^{(1)}} \right) = y$, we obtain:

$$\begin{aligned} \psi_{N,1} \left(\lambda - \frac{v}{M_N^{(1)}} \right) &= \int_{\mathbb{R}} |H_N(y)|^\alpha \phi \left(\lambda - \frac{v}{M_N^{(1)}} + \frac{y}{\tau} \right) dy \\ &= \sum_{j \in \mathbb{Z}} \int_{(2j-1)\pi}^{(2j+1)\pi} |H_N(y)|^\alpha \phi \left(\lambda - \frac{v}{M_N^{(1)}} + \frac{y}{\tau} \right) dy. \end{aligned}$$

(8)

Let $y - 2j\pi = s$. Since $|H_N(\cdot)|^\alpha$ is 2π -periodic function, we get

$$\psi_{N,1} \left(\lambda - \frac{v}{M_N^{(1)}} \right) = \sum_{j \in \mathbb{Z}} \int_{-\pi}^{+\pi} |H_N(s)|^\alpha \phi \left(\lambda - \frac{v}{M_N^{(1)}} + \frac{s}{\tau} + \frac{2\pi j}{\tau} \right) ds.$$

Since the function ϕ is uniformly continuous on $[-\Omega, \Omega]$ and the fact that $|H_N|^\alpha$ is a kernel,

the right hand side of the last equality converges to $\sum_{j \in \mathbb{Z}} \phi \left(\lambda + \frac{2\pi j}{\tau} \right)$. Let j be an integer

such that $-\Omega < \frac{\tau\lambda + 2\pi j}{\tau} < \Omega$. The definition of τ implies that $|\tau\lambda| < \tau\Omega < \pi$. It is easy to

see that $|j| < 1$ and then $j = 0$. Since $H_N^{(1)}$ is a kernel, we obtain that $\psi_{N,1}\left(\lambda - \frac{v}{M_N^{(1)}}\right)$

converges to $\phi(\lambda)$. On the other hand, $\psi_{N,2}\left(\lambda - \frac{v}{M_N^{(1)}}\right) = \sum_{i=1}^q \left| H_N^{(1)}\left(w_i - \tau\left(\lambda - \frac{v}{M_N^{(1)}}\right)\right) \right|^\alpha$

Since w_i is different from $\tau\lambda$ and from the lemma 2.2, we get

$$\psi_{N,2}\left(\lambda - \frac{v}{M_N^{(1)}}\right) \leq \left[2\pi \left(\frac{2}{\pi}\right)^{2k\alpha} n^{2k\alpha-1} \right]^{-1} \frac{1}{cte} \sum_{i=1}^q c_i, \text{ where } cte = \inf \left| \sin \left(\frac{w_i - \tau\left(\lambda - \frac{v}{M_N^{(1)}}\right)}{2} \right) \right|^\alpha.$$

Therefore, $\psi_{N,2}\left(\lambda - \frac{v}{M_N^{(1)}}\right) = O\left(\frac{1}{n^{2k\alpha-1}}\right)$. Thus, we have $E[f_N(\lambda)] - [\phi(\lambda)]^\alpha = o(1)$.

The rate of convergence: We assume that the spectral density ϕ satisfies the hypothesis H . We denote by $F = |Bias(f_N(\lambda))| = |E[f_N(\lambda)] - [\phi(\lambda)]^\alpha|$. It follows that

$$F \leq \frac{p}{2\alpha} \int_{-1}^1 W(v) \left[\left| \psi_N\left(\lambda - \frac{v}{M_N^{(1)}}\right) \right|^\alpha + [\phi(\lambda)]^\alpha \right] \left| \psi_N\left(\lambda - \frac{v}{M_N^{(1)}}\right) - \phi(\lambda) \right| dv.$$

Since $\psi_N\left(\lambda - \frac{v}{M_N^{(1)}}\right)$ converges to $\phi(\lambda)$, getting the rate of the convergence for F requires to

examine the rate of convergence of $\int_{-1}^1 W(v) \left| \psi_N\left(\lambda - \frac{v}{M_N^{(1)}}\right) - \phi(\lambda) \right| dv$. Indeed, from (3), we

obtain $\psi_N\left(\lambda - \frac{v}{M_N^{(1)}}\right) = \int_{-\pi}^{\pi} |H_N\left(y - \tau\lambda + \frac{\tau v}{M_N^{(1)}}\right)|^\alpha \phi\left(\frac{y}{\tau}\right) dy$.

Denote by $\Delta(\psi_N, \phi) = \psi_N\left(\lambda - \frac{v}{M_N^{(1)}}\right) - \phi(\lambda)$. Putting $t = -\left(y - \tau\lambda + \frac{\tau v}{M_N^{(1)}}\right)$ and using the

$$|\Delta(\psi_N, \phi)| \leq C_1 \int_{\tau\lambda - \frac{\tau v}{M_N^{(1)}} - \pi}^{\tau\lambda - \frac{\tau v}{M_N^{(1)}} + \pi} |H_N(t)|^\alpha \left| \frac{v}{M_N^{(1)}} + \frac{t}{\tau} \right|^\gamma dt.$$

condition H , we get

It is easy to show that

$$\int_{-1}^1 W(v) |\Delta(\psi_N, \phi)| dv \leq 2^\gamma C_1 \left| \frac{1}{M_N^{(1)}} \right|^\gamma \int_{-1}^1 W(v) |v|^\gamma dv$$

$$+ 2^\gamma \frac{C_1}{\tau^\gamma} \int_{-1}^1 W(v) \int_{\tau_1 \lambda - \frac{\tau v}{M_N^{(1)}} - \pi}^{\tau \lambda - \frac{\tau v}{M_N^{(1)}} + \pi} |H_N(t)|^\alpha |t|^\gamma dt dv$$

The second integral of the right hand side is bounded as follows:

$$\int_{\tau \lambda - \frac{\tau v}{M_N^{(1)}} - \pi}^{\tau \lambda - \frac{\tau v}{M_N^{(1)}} + \pi} |H_N(t)|^\alpha |t|^\gamma dt \leq \int_{-|\tau \lambda| - \frac{\tau v}{M_N^{(1)}} - \pi}^{-\pi} |H_N(t)|^\alpha |t|^\gamma dt$$

$$+ \int_{-\pi}^{\pi} |H_N(t)|^\alpha |t|^\gamma dt$$

$$+ \int_{\pi}^{|\tau \lambda| + \frac{\tau v}{M_N^{(1)}} + \pi} |H_N(t)|^\alpha |t|^\gamma dt.$$

(9)

The function $|H_N(\cdot)|$ is even, then the first and the last integrals in the right hand side of (9) are

equal. Since $\frac{\tau v}{M_N}$ converges to zero and $\tau \lambda < \tau \Omega < \pi$, for a large N we have:

$$\int_{\pi}^{|\tau \lambda| + \frac{\tau v}{M_N^{(1)}} + \pi} |H_N(t)|^\alpha |t|^\gamma dt \leq (2\pi)^\gamma \int_{\pi}^{|\tau \lambda| + \frac{\tau v}{M_N^{(1)}} + \pi} |H_N(t)|^\alpha dt$$

$$\leq \frac{(2\pi)^\gamma}{B_{\alpha, N}} \frac{|\tau \lambda| + \frac{\tau}{M_N^{(1)}}}{\left| \sin \left(\pi + |\tau \lambda| + \frac{\tau}{M_N^{(1)}} \right) \right|^{2k\alpha}}$$

From the lemma 2.1, we obtain,

$$\int_{\pi}^{|\tau \lambda| + \frac{\tau v}{M_N^{(1)}} + \pi} |H_N(t)|^\alpha |t|^\gamma dt = \begin{cases} \mathcal{O} \left(\frac{1}{n^{2k\alpha-1}} \right) & \text{if } \lambda \neq 0, \\ \mathcal{O} \left(\frac{1}{M_N^{(1)} n^{2k\alpha-1}} \right) & \text{if } \lambda = 0, \end{cases}$$

Thus, the result follows.

Theorem 3.2. Let λ a real number belonging to $]-\Omega, \Omega[$, and $\tau\lambda = w_i$. Choose k such that

$$\lim_{N \rightarrow \infty} \frac{(M_N^{(1)})^{2k\alpha}}{n^{2k\alpha-1}} = 0$$

. Then,

i) $E[f_N(\lambda)] - [\phi(\lambda)]^{\frac{p}{\alpha}} = o(1)$

ii) If ϕ satisfies the hypothesis $|\phi(x) - \phi(y)| \leq cste |x - y|^{-\gamma}$, with $\frac{\gamma+1}{2k} < \alpha < 2$, then

$$\left| E[f_N(\lambda)] - [\phi(\lambda)]^{\frac{p}{\alpha}} \right| = \begin{cases} O\left(\frac{1}{(M_N^{(2)})^\gamma} + \frac{(M_N^{(1)})^{2k\alpha}}{n^{2k\alpha-1}} \right) & \text{if } 0 < \gamma \leq 1 \\ O\left(\frac{1}{M_N^{(2)}} + \frac{(M_N^{(1)})^{2k\alpha}}{n^{2k\alpha-1}} \right) & \text{if } 1 < \gamma \leq 2 \end{cases}$$

Proof :

From the definition of the estimator, we have

$$E[f_N(\lambda)] = \int_{-\pi}^{\pi} \frac{W_N^{(2)}(\lambda - u) - \frac{M_N^{(2)}}{M_N^{(1)}} W_N^{(1)}(\lambda - u)}{1 - \frac{M_N^{(2)}}{M_N^{(1)}}} [\psi_N(u)]^{\frac{p}{\alpha}} du$$

$$E[f_N(\lambda)] = \int_{-\pi}^{\lambda - \frac{1}{M_N^{(1)}}} + \int_{\lambda - \frac{1}{M_N^{(1)}}}^{\lambda + \frac{1}{M_N^{(1)}}} + \int_{\lambda + \frac{1}{M_N^{(1)}}}^{\pi} = E_1 + E_2 + E_3$$

$$\lambda - u = v, \quad E_2 = \int_{-\frac{1}{M_N^{(1)}}}^{\frac{1}{M_N^{(1)}}} \frac{M_N^{(2)} [W[M_n^{(2)}v] - W[M_N^{(1)}v]]}{1 - \frac{M_N^{(2)}}{M_N^{(1)}}} [\psi_N(\lambda - v)]^{\frac{p}{\alpha}} dv.$$

Put

for a large N.

Therefore $E_2 = 0$

$$E_1 = \frac{M_N^{(2)}}{1 - \frac{M_N^{(2)}}{M_N^{(1)}}} \int_{-\pi}^{\lambda - \frac{1}{M_N^{(1)}}} W[M_N^{(2)}(\lambda - u)] [\psi_N(u)]^{\frac{p}{\alpha}} du - \frac{M_N^{(2)}}{1 - \frac{M_N^{(2)}}{M_N^{(1)}}} \int_{-\pi}^{\lambda - \frac{1}{M_N^{(1)}}} W[M_N^{(1)}(\lambda - u)] [\psi_N(u)]^{\frac{p}{\alpha}} du$$

Put $M_N^{(2)}(\lambda - u) = v$ in the first integral and put $M_N^{(1)}(\lambda - u) = w$, in the second integral and for a large N , we have $M_N^{(1)}(\lambda + \pi) > 1$ et $M_N^{(2)}(\lambda + \pi) > 1$. As W is null outside of $[-1, 1]$, for large N , the second integral of E_1 is zero. Therefore,

$$E_1 = \frac{1}{1 - \frac{M_N^{(2)}}{M_N^{(1)}}} \int_{\frac{M_N^{(2)}}{M_N^{(1)}}}^1 W(v) \left[\psi_N \left(\lambda - \frac{v}{M_N^{(2)}} \right) \right]^{\frac{p}{\alpha}} dv. \tag{10}$$

$$E_3 = \frac{M_N^{(2)}}{1 - \frac{M_N^{(2)}}{M_N^{(1)}}} \int_{\lambda + \frac{1}{M_N^{(1)}}}^{\pi} W[M_N^{(2)}(\lambda - u)] [\psi_N(u)]^{\frac{p}{\alpha}} du - \frac{M_N^{(2)}}{1 - \frac{M_N^{(2)}}{M_N^{(1)}}} \int_{\lambda + \frac{1}{M_N^{(1)}}}^{\pi} W[M_N^{(1)}(\lambda - u)] [\psi_N(u)]^{\frac{p}{\alpha}} du$$

Putting $M_N^{(2)}(\lambda - u) = v$ in the first integral and $M_N^{(1)}(\lambda - u) = w$ in the second integral, we obtain

$$E_3 = \frac{1}{1 - \frac{M_N^{(2)}}{M_N^{(1)}}} \int_{\frac{M_N^{(2)}}{M_N^{(1)}}}^{\frac{M_N^{(2)}}{M_N^{(1)}(\lambda - \pi)}} W(v) \left[\psi_N \left(\lambda - \frac{v}{M_N^{(2)}} \right) \right]^{\frac{p}{\alpha}} dv - \frac{M_N^{(2)}}{1 - \frac{M_N^{(2)}}{M_N^{(1)}}} \int_{M_N^{(1)}(\lambda - \pi)}^{-1} W(w) \left[\psi_N \left(\lambda - \frac{w}{M_N^{(1)}} \right) \right]^{\frac{p}{\alpha}} dw.$$

For large N we have, $M_N^{(2)}(\lambda - \pi) < -1$ and $M_N^{(1)}(\lambda - \pi) < -1$.

$$E_3 = \frac{1}{1 - \frac{M_N^{(2)}}{M_N^{(1)}}} \int_{\frac{M_N^{(2)}}{M_N^{(1)}}}^1 W(v) \left[\psi_N \left(\lambda + \frac{v}{M_N^{(2)}} \right) \right]^{\frac{p}{\alpha}} dv. \tag{11}$$

It is easy to show that for a large N

$$\frac{1}{1 - \frac{M_N^{(2)}}{M_N^{(1)}}} \int_{\frac{M_N^{(2)}}{M_N^{(1)}}}^1 W(v) dv = \frac{1}{2}. \quad (12)$$

$$\left| E[f_N(\lambda)] - [\phi(\lambda)]^\alpha \right| = \left| E_1 + E_3 - [\phi(\lambda)]^\alpha \right| \leq \left| E_1 - \frac{1}{2} [\phi(\lambda)]^\alpha \right| + \left| E_3 - \frac{1}{2} [\phi(\lambda)]^\alpha \right|$$

From (10) and (12), for a large N , we have

$$\left| E_1 - \frac{1}{2} [\phi(\lambda)]^\alpha \right| \leq \frac{1}{1 - \frac{M_N^{(2)}}{M_N^{(1)}}} \int_{\frac{M_N^{(2)}}{M_N^{(1)}}}^1 W(v) \left| \psi_{N,1} \left(\lambda - \frac{v}{M_N^{(2)}} \right) + \psi_{N,2} \left(\lambda - \frac{v}{M_N^{(2)}} \right) - \phi(\lambda) \right|^\alpha dv$$

As $\frac{p}{\alpha} < 1$, we obtain

$$\begin{aligned} \left| E_1 - \frac{1}{2} [\phi(\lambda)]^\alpha \right| &\leq \frac{1}{1 - \frac{M_N^{(2)}}{M_N^{(1)}}} \int_{\frac{M_N^{(2)}}{M_N^{(1)}}}^1 W(v) \left| \psi_{N,1} \left(\lambda - \frac{v}{M_N^{(2)}} \right) - \phi(\lambda) \right|^\alpha dv \\ &\quad + \frac{1}{1 - \frac{M_N^{(2)}}{M_N^{(1)}}} \int_{\frac{M_N^{(2)}}{M_N^{(1)}}}^1 W(v) \left| \psi_{N,2} \left(\lambda - \frac{v}{M_N^{(2)}} \right) \right|^\alpha dv. \end{aligned}$$

On the other hand,

$$\int_{\frac{M_N^{(2)}}{M_N^{(1)}}}^1 W(v) \left| \psi_{N,1} \left(\lambda \pm \frac{v}{M_N^{(2)}} \right) - \phi(\lambda) \right|^\alpha dv \leq \int_0^1 W(v) \left| \psi_{N,1} \left(\lambda \pm \frac{v}{M_N^{(2)}} \right) - \phi(\lambda) \right|^\alpha dv$$

For all λ belonging to $]-\pi, \pi[$ $\psi_{N,1} \left(\lambda \pm \frac{v}{M_N^{(2)}} \right)$ converges to $\phi(\lambda)$, uniformly in

$v \in [-1, 1]$. Therefore, $\frac{1}{1 - \frac{M_N^{(2)}}{M_N^{(1)}}} \int_{\frac{M_N^{(2)}}{M_N^{(1)}}}^1 W(v) \left| \psi_{N,1} \left(\lambda \pm \frac{v}{M_N^{(2)}} \right) - \phi(\lambda) \right|^\alpha dv$ converge to zero.

$$\psi_{N,2} \left(\lambda_i \pm \frac{v}{M_N^{(2)}} \right) \leq \sum_{m=1}^q \frac{a_m}{B'_{\alpha,N}} \frac{1}{\left| \sin \left[\frac{1}{2} \left(w_i \pm \frac{\tau v}{M_N^{(2)}} - w_m \right) \right] \right|^{2k\alpha}}$$

Since $\tau\lambda = w_i$,

$$+ \frac{a_i}{B'_{\alpha,N}} \frac{1}{\left| \sin \left[\frac{1}{2} \left(\pm \frac{\tau v}{M_N^{(2)}} \right) \right] \right|^{2k\alpha}}.$$

$$\limsup_{N \rightarrow +\infty, v \in [-1,1]} \frac{1}{\left| \sin \left[\frac{1}{2} \left(w_i \pm \frac{\tau v}{M_N^{(2)}} - w_m \right) \right] \right|^{2k\alpha}} = \frac{1}{\left| \sin \left[\frac{1}{2} (w_i - w_m) \right] \right|^{2k\alpha}}.$$

For all $m \neq i$,

Thus, for large N, we get

$$\sum_{m=1}^q \frac{a_m}{B'_{\alpha,N}} \frac{1}{\left| \sin \left[\frac{1}{2} \left(w_i \pm \frac{\tau v}{M_N^{(2)}} - w_m \right) \right] \right|^{2k\alpha}} \leq \left(\varepsilon + \frac{1}{\inf_{m \in \{1,2,\dots,q\} - \{i\}} \left| \sin \left[\frac{1}{2} (w_i - w_m) \right] \right|^{2k\alpha}} \right) \sum_{m=1}^q \frac{a_m}{B'_{\alpha,N}}.$$

$$\sum_{m=1}^q \frac{a_m}{B'_{\alpha,N}} \frac{1}{\left| \sin \left[\frac{1}{2} \left(w_i \pm \frac{\tau v}{M_N^{(2)}} - w_m \right) \right] \right|^{2k\alpha}} = O\left(\frac{1}{n^{2k\alpha-1}} \right).$$

From the lemma 2.1, we obtain

(13)

$$\sup_{v \in [0,1]} \left| \frac{\tau v}{M_N^{(2)}} \right| = \frac{\tau}{M_N^{(2)}} < \pi$$

On the other hand, for large N, we have

. Consequently

$$\frac{a_i}{B'_{\alpha,N}} \frac{1}{\left| \sin \left[\frac{1}{2} \left(\pm \frac{\tau v}{M_N^{(2)}} \right) \right] \right|^{2k\alpha}} \leq \frac{a_i}{B'_{\alpha,N}} \frac{\pi^{2k\alpha}}{\left| \frac{\tau v}{M_N^{(2)}} \right|^{2k\alpha}}.$$

As

$$\frac{\tau}{M_N^{(1)}} \leq \frac{\tau v}{M_N^{(2)}} \leq \frac{\tau}{M_N^{(2)}}, \text{ we obtain}$$

$$\frac{a_i}{B'_{\alpha,N}} \frac{1}{\left| \sin \left[\frac{1}{2} \left(\pm \frac{\tau v}{M_N^{(2)}} \right) \right] \right|^{2k\alpha}} \leq \frac{a_i}{B'_{\alpha,N}} \frac{\pi^{2k\alpha}}{\left| \frac{\tau}{M_N^{(1)}} \right|^{2k\alpha}}.$$

The lemma 2.2 gives

$$\frac{a_i}{B'_{\alpha,N}} \frac{1}{\left| \sin \left[\frac{1}{2} \left(+ - \frac{\tau v}{M_N^{(2)}} \right) \right] \right|^{2k\alpha}} = O \left(\frac{\left(M_N^{(1)} \right)^{2k\alpha}}{n^{2k\alpha-1}} \right). \quad (14)$$

Thus, we get

$$\frac{2^\alpha}{1 - \frac{M_N^{(2)}}{M_N^{(1)}}} \int_{\frac{M_N^{(2)}}{M_N^{(1)}}}^1 W(v) \left| \psi_{N,2} \left(\tau \lambda \pm \frac{\tau v}{M_N^{(2)}} \right) \right|^\alpha dv = O \left[\frac{1}{n^{(2k\alpha-1)\frac{p}{\alpha}}} + \left(\frac{\left(M_N^{(1)} \right)^{2k\alpha}}{n^{2k\alpha-1}} \right)^\alpha \right] \quad (15)$$

Choosing $M_N^{(1)}$ such that $\frac{\left(M_N^{(1)} \right)^{2k\alpha}}{n^{2k\alpha-1}}$ converges to 0. For example $M_N^{(1)} = n^b$ with $0 < b < 1 - \frac{1}{2k\alpha}$. Thus, $\lim_{N \rightarrow +\infty} E[f_N(\lambda)] - [\phi(\lambda)]^\alpha = 0$.

Theorem 4.2 Let $-\Omega < \lambda < \Omega$ such that $\phi(\lambda) > 0$. Then, $\text{var}[f_N(\lambda)]$ converges to zero. If $M_N^{(1)} = n^c$ with $\frac{1}{2k^2\alpha^2} < c < \frac{1}{2}$, then $\text{var}[f_N(\lambda)] = O\left(\frac{1}{n^{(1-2c)}}.$

Proof Consider the case where $\tau\lambda \notin \{w_1, w_2, \dots, w_q\}$. It is clear that the variance of f_N can be written as follows:

$$\text{var}[f_N(\lambda)] = \int_{\mathbb{R}^2} W_N^{(1)}(\lambda - u) W_N^{(1)}(\lambda_1 - u') \text{cov}[\hat{I}_N(u), \hat{I}_N(u')] dud u'.$$

By using the fact that W is zero for $|\lambda| > 1$, for large N , we get

$$\text{var}[f_N(\lambda)] = \int_{-1}^1 \text{cov} \left[\hat{I}_N \left(\lambda - \frac{x_1}{M_N} \right), \hat{I}_N \left(\lambda - \frac{x'_1}{M_N} \right) \right] W(x_1) W(x'_1) dx_1 dx'_1.$$

We define two subsets of the $[-1, 1]^2$ by:

- $L_1 = \{(x_1, x'_1) \in [-1, 1]^2; |x_1 - x'_1| > \sigma_N\}$,
- $L_2 = \{(x_1, x'_1) \in [-1, 1]^2; |x_1 - x'_1| \leq \sigma_N\}$,

where σ_N is a nonnegative real, converging to 0. We split the integral into an integral over the sub region L_2 and an integral over L_1 : $\text{var}[f_N(\lambda)] = \int_{L_2} + \int_{L_1} \stackrel{\Delta}{=} J_1 + J_2$.

By Cauchy-Schwartz inequality and theorem 3.1, we obtain

$$J_1 \leq C \int_{|x_1 - x'_1| \leq \sigma_N} W(x_1)W(x'_1)dx_1dx'_1. \quad \text{where } C \text{ is constant. Thus, we obtain}$$

$$J_1 = O(\sigma_N) \quad (16)$$

It remains to show that J_2 converges to zero. For simplicity, we define

$$\lambda_1 = \lambda - \frac{x_1}{M_N^{(1)}}; \quad \lambda_2 = \lambda - \frac{x'_1}{M_N^{(1)}}, \quad \text{and} \quad C(\lambda) = \text{cov} \left[\hat{I}_N \left(\lambda - \frac{x_1}{M_N^{(1)}} \right), \hat{I}_N \left(\lambda - \frac{x'_1}{M_N^{(1)}} \right) \right].$$

We first show that $C(\lambda)$ converges to zero uniformly in $x_1, x'_1 \in [-1, 1]$. Indeed, from lemma 2.3, we have

$$E\hat{I}_N(v) - \hat{I}_N(v) = F_{p,\alpha}^{-1}[C_\alpha]^{-p/\alpha} \int_{-\infty}^{\infty} \frac{\text{Re}(e^{iuI_N(v)}) - e^{-C_\alpha|u|^\alpha \psi_N(v)}}{|u|^{1+p}} du.$$

Thus, the expression of the covariance becomes

$$C(\lambda) = F_{p,\alpha}^{-2} C_\alpha^{-\frac{2p}{\alpha}} \int_{R^2} E \left[\prod_{k=1}^2 \cos(u_k I_N(\lambda_k)) \right] - \exp \left\{ -C_\alpha \sum_{k=1}^2 |u_k|^\alpha \psi_N(\lambda_k) \right\} \frac{du_1 du_2}{|u_1 u_2|^{1+p}}$$

The following equality: $2 \cos x \cos y = \cos(x+y) + \cos(x-y)$, implies that

$$E \left[\prod_{k=1}^2 (\cos u_k I_N(\lambda_k)) \right] = \frac{1}{2} \exp \left[-C_\alpha \int \left| (\tau)^\frac{1}{\alpha} \sum_{k=1}^2 u_k H_N(\tau \lambda_k - \tau v) \right|^\alpha d\mu(v) \right] + \frac{1}{2} \exp \left[-C_\alpha \int \left| (\tau)^\frac{1}{\alpha} \sum_{k=1}^2 (-1)^{k+1} u_k H_N(\tau \lambda_k - \tau v) \right|^\alpha d\mu(v) \right].$$

By substituting the expression for $C(\lambda)$ and changing the variable u_2 to $(-u_2)$ in the second term, we obtain

$$C(\lambda) = F_{p,\alpha}^{-2} C_\alpha^{-\frac{2p}{\alpha}} \int_{\mathbb{R}^2} \left(e^{-K} - e^{-K'} \right) \frac{du_1 du_2}{|u_1 u_2|^{1+p}}, \quad (17)$$

where
$$K = C_\alpha \int_{\mathbb{R}} \left| (\tau)^{\frac{1}{\alpha}} \sum_{k=1}^2 u_k H_N(\tau \lambda_k - \tau v) \right|^\alpha d\xi(v)$$
 and

$$K' = C_\alpha \tau \sum_{k=1}^2 |u_k|^\alpha \int_{\mathbb{R}} |H_N(\tau \lambda_k - \tau v)|^\alpha \xi(v) dv$$

Since $K, K' > 0$, $\left| e^{-K} - e^{-K'} \right| \leq |K - K'| \exp\{|K - K'| - K'\}$, we obtain:

$$|K - K'| \leq 2C_\alpha \tau |u|^\alpha Q_N(\lambda_1; \lambda_2), \quad \text{where}$$

$$Q_N(\lambda_1; \lambda_2) = \int_{-\Omega}^{\Omega} |H_N(\tau \lambda_1 - \tau u)|^\alpha |H_N(\tau \lambda_2 - \tau u)|^\alpha d\xi(u)$$

Now, let us show that $Q_N(\lambda_1; \lambda_2)$ converges to zero. Indeed, since ϕ is bounded on $[-\Omega, \Omega]$, we have

$$\begin{aligned} Q_N(\lambda_1; \lambda_2) &\leq \sup(\phi) \int_{-\Omega}^{\Omega} |H_N(\tau \lambda_1 - \tau u) H_N(\tau \lambda_2 - \tau u)|^\alpha du \\ &\quad + \sum_{i=1}^q c_i |H_N(\tau \lambda_1 - \tau w_i) H_N(\tau \lambda_2 - \tau w_i)|^\alpha \end{aligned} \quad (18)$$

From the definition of H_N , we can write

$$\int_{-\Omega}^{\Omega} |H_N(\tau \lambda_1 - \tau v) H_N(\tau \lambda_2 - \tau v)|^\alpha dv = \int_{-\Omega}^{\Omega} \frac{1}{B_{\alpha,N}'} \left| \frac{\sin \left[\frac{n}{2} (\tau \lambda_1 - \tau v) \right]}{\sin \left[\frac{1}{2} (\tau \lambda_1 - \tau v) \right]} \right|^{k\alpha} \left| \frac{\sin \left[\frac{n}{2} (\tau \lambda_2 - \tau v) \right]}{\sin \left[\frac{1}{2} (\tau \lambda_2 - \tau v) \right]} \right|^{k\alpha} dv.$$

a) First step:

We show that the denominators of the first and second terms under the last integral do not vanish for the same v , so we suppose that v exists, belonging to $[-\Omega, \Omega]$ and $z, z' \in \mathbb{Z}$ such as: $\tau \lambda_1 - \tau v = 2z\pi$ and $\tau \lambda_2 - \tau v = 2z'\pi$. Since $\lambda_1 \neq \lambda_2$, then z and z' are different.

Therefore, $z - z' = \frac{\tau}{2\pi} (\lambda_1 - \lambda_2)$. Hence, $|z - z'| = \frac{1}{w} |\lambda_1 - \lambda_2|$. As $\lim_{N \rightarrow \infty} |\lambda_1 - \lambda_2| = 0$,

consequently, for a large N we get: $|z - z'| < \frac{1}{2}$.

Thus, we obtain a contradiction with the fact that z and z' are different integers.

b) Second step

We assume there exist q points, $V_1, V_2, \dots, V_q \in [-\Omega, \Omega]$ such as:

for $j = 1, 2, \dots, q$ $\tau\lambda_1 - \tau V_j \in 2\pi Z$ then $\frac{\lambda_1 - V_j}{w} \in Z$, and we assume that there exist q'

points $V'_1, V'_2, \dots, V'_{q'} \in [-\Omega, \Omega]$ such that for $i = 1, 2, \dots, q'$ $\frac{\lambda_2 - V'_i}{w} \in Z$. Showing that,

$|V_j| \leq \Omega$, $|V'_i| \leq \Omega$ for $1 \leq j \leq q$ and $1 \leq i \leq q'$. Indeed, $-1 < \frac{\lambda - \Omega}{w} < 0$ and

$0 < \frac{\lambda + \Omega}{w} < 1$ because $w > 2\Omega$. Hence $\frac{\lambda - \Omega}{w} \notin Z$ and $\frac{\lambda + \Omega}{w} \notin Z$. For a large N , we get

that $\left[\frac{\lambda - \Omega}{w} \right]_E < \frac{\lambda_1 - \Omega}{w} < 1 + \left[\frac{\lambda - \Omega}{w} \right]_E$, where $[x]_E$ is the integer part of x . Hence,

$\frac{\lambda_1 - \Omega}{w} \notin Z$. In the same manner, we show that $\frac{\lambda_1 + \Omega}{w} \notin Z$. Similarly, it can be shown that

$\frac{\lambda_2 + \Omega}{w} \notin Z$. Thus, $|V_j| \leq \Omega$ and $|V'_i| \leq \Omega$.

c) Third step

We classify V_j and V'_i by increasing order: $-\Omega < V_{j_1} < V_{j_2} < \dots < V_{j_{q+q'}} < \Omega$, and we write the integral in the following manner:

$$\int_{-\Omega}^{\Omega} \left| \frac{\sin \left[\frac{n}{2} (\tau\lambda_1 - \tau v) \right]}{\sin \left[\frac{1}{2} (\tau\lambda_1 - \tau v) \right]} \right|^{k\alpha} \left| \frac{\sin \left[\frac{n}{2} (\tau\lambda_2 - \tau v) \right]}{\sin \left[\frac{1}{2} (\tau\lambda_2 - \tau v) \right]} \right|^{k\alpha} dv = I_1 + \sum_{i=1}^{q+q'} I_{2,i} + \sum_{i=1}^{q+q'-1} I_{3,i} + I_4$$

where

$$\begin{aligned}
I_1 &= \int_{-\Omega}^{V_{j_1} - \delta(N)} \left| \frac{\sin \left[\frac{n}{2} (\tau \lambda_1 - \tau v) \right]}{\sin \left[\frac{1}{2} (\tau \lambda_1 - \tau v) \right]} \right|^{k\alpha} \left| \frac{\sin \left[\frac{n}{2} (\tau \lambda_2 - \tau v) \right]}{\sin \left[\frac{1}{2} (\tau \lambda_2 - \tau v) \right]} \right|^{k\alpha} dv \\
I_{2,i} &= \int_{V_{j_i} - \delta(N)}^{V_{j_i} + \delta(N)} \left| \frac{\sin \left[\frac{n}{2} (\tau \lambda_1 - \tau v) \right]}{\sin \left[\frac{1}{2} (\tau \lambda_1 - \tau v) \right]} \right|^{k\alpha} \left| \frac{\sin \left[\frac{n}{2} (\tau \lambda_2 - \tau v) \right]}{\sin \left[\frac{1}{2} (\tau \lambda_2 - \tau v) \right]} \right|^{k\alpha} dv \\
I_{3,i} &= \int_{V_{j_i} + \delta(N)}^{V_{j_{i+1}} - \delta(N)} \left| \frac{\sin \left[\frac{n}{2} (\tau \lambda_1 - \tau v) \right]}{\sin \left[\frac{1}{2} (\tau \lambda_1 - \tau v) \right]} \right|^{k\alpha} \left| \frac{\sin \left[\frac{n}{2} (\tau \lambda_2 - \tau v) \right]}{\sin \left[\frac{1}{2} (\tau \lambda_2 - \tau v) \right]} \right|^{k\alpha} dv \\
I_4 &= \int_{V_{j_{q+q'}} + \delta(N)}^{\Omega} \left| \frac{\sin \left[\frac{n}{2} (\tau \lambda_1 - \tau v) \right]}{\sin \left[\frac{1}{2} (\tau \lambda_1 - \tau v) \right]} \right|^{k\alpha} \left| \frac{\sin \left[\frac{n}{2} (\tau \lambda_2 - \tau v) \right]}{\sin \left[\frac{1}{2} (\tau \lambda_2 - \tau v) \right]} \right|^{k\alpha} dv
\end{aligned}$$

where $\delta(N)$ is a nonnegative real number converging to zero and satisfying:

$$-\Omega < V_{j_1} - \delta(N) < V_{j_1} + \delta(N) < V_{j_2} - \delta(N) < V_{j_2} + \delta(N) < \dots < V_{j_{q+q'}} - \delta(N) < V_{j_{q+q'}} + \delta(N) < \Omega,$$

$$\delta(N) < \left| \frac{\lambda_1 - \lambda_2}{2} \right|.$$

and

First, we show that the first integral converges to zero. We know that for a large N , we have $\lambda_1 < \Omega$. For simplicity without loss of generality, we assume that for all i , $\tau \lambda_1 - \tau V_{j_i} = 2k_i \pi$ with $k_i \in \mathbb{Z}$. Similarly, we can show the result if we rather assume that $\tau \lambda_2 - \tau V_{j_i} = 2k_i \pi$ with $k_i \in \mathbb{Z}$. Since there is no v between $-\Omega$ and $V_{j_1} - \delta(N)$ on which the denominators are vanishing,

$$I_1 \leq \frac{V_{j_1} - \delta(N) + \Omega}{\inf \left[\left| \sin \frac{\tau \delta(N)}{2} \right|^{k\alpha}, \left| \sin \frac{\tau (\lambda_1 + \Omega)}{2} \right|^{k\alpha} \right]} \frac{1}{\inf \left[\left| \sin \frac{\tau (\lambda_2 - V_{j_1} + \delta(N))}{2} \right|^{k\alpha}, \left| \sin \frac{\tau (\lambda_2 + \Omega)}{2} \right|^{k\alpha} \right]}.$$

By substituting for V_{j_1} in the last inequality, we obtain

$$\left| \sin \frac{\tau (\lambda_2 - V_{j_1} + \delta(N))}{2} \right|^{k\alpha} = \left| \sin \frac{\tau |\lambda_2 - \lambda_1 + \delta(N)|}{2} \right|^{k\alpha}. \text{ For a large } N, \text{ we have}$$

$\frac{\tau |\lambda_2 - \lambda_1 + \delta(N)|}{2} \leq \tau\Omega + \frac{\tau\delta(N)}{2} < \pi - \frac{\tau\delta(N)}{2}$. On the other hand, two cases are possible:

1. $\lambda_2 - \lambda_1 > 0$, then we have $|\lambda_2 - \lambda_1 + \delta(N)| = \lambda_2 - \lambda_1 + \delta(N) > \delta(N)$
2. $\lambda_2 - \lambda_1 < 0$, since $|\lambda_2 - \lambda_1| > 2\delta(N)$, we have

$$|\lambda_2 - \lambda_1 + \delta(N)| = \lambda_1 - \lambda_2 - \delta(N) > \delta(N).$$

Therefore, $\frac{\tau\delta(N)}{2} < \frac{\tau |\lambda_2 - \lambda_1 + \delta(N)|}{2} < \pi - \frac{\tau\delta(N)}{2}$. For a large N , we have $\tau\delta(N) < 2\pi - \tau(\lambda_1 + \Omega)$ and $\tau\delta(N) < 2\pi - \tau(\lambda_2 + \Omega)$. Then,

$$\frac{\tau\delta(N)}{2} < \frac{\tau(\lambda_1 + \Omega)}{2} < \pi - \frac{\tau\delta(N)}{2} \quad \text{and} \quad \frac{\tau\delta(N)}{2} < \frac{\tau(\lambda_2 + \Omega)}{2} < \pi - \frac{\tau\delta(N)}{2}.$$

Consequently,

$$I_1 \leq \frac{V_{j_i} - \delta(N) + \Omega}{\left| \sin \frac{\tau\delta(N)}{2} \right|^{2k\alpha}}.$$

For the integral $I_{2,i}$, we bound the first fraction under integral by $n^{k\alpha}$:

$$I_{2,i} \leq n^{k\alpha} \int_{V_{j_i} - \delta(N)}^{V_{j_i} + \delta(N)} \frac{1}{\left| \sin \left[\frac{1}{2} (\tau\lambda_2 - \tau v) \right] \right|^{k\alpha}} dv.$$

. By substituting for V_{j_i} in the last inequality and

putting $v = u - \frac{2k\pi}{\tau}$, we get

$$I_{2,i} \leq n^{k\alpha} \int_{\lambda_1 - \delta(N)}^{\lambda_1 + \delta(N)} \frac{1}{\left| \sin \left[\frac{1}{2} (\tau\lambda_2 - \tau u) \right] \right|^{k\alpha}} du.$$

. Since $|\lambda_1 - u| < \delta(N)$, it is easy to note that :

$$|\lambda_2 - u| \geq |\lambda_2 - \lambda_1| - |\lambda_1 - u| \geq |\lambda_2 - \lambda_1| - \delta(N) > \frac{|\lambda_2 - \lambda_1|}{2}$$

Since $\delta(N)$ converges to zero, for a large N we have $\delta(N) < \frac{2}{\tau} \left(\pi - \frac{\tau}{2} |\lambda_2 - \lambda_1| \right)$, therefore

$$0 < \tau \frac{|\lambda_2 - \lambda_1|}{4} < \tau \frac{|\lambda_2 - u|}{2} < \tau \frac{|\lambda_2 - \lambda_1| + \delta(N)}{2} < \pi.$$

Consequently:

$$I_{2,i} \leq \frac{2\delta(N)n^{k\alpha}}{\inf \left[\sin \frac{|\lambda_1 - \lambda_2| \tau}{4}; \sin \frac{|\lambda_1 - \lambda_2| \tau + \delta(N)\tau}{2} \right]},$$

where $|\lambda_2 - \lambda_1| = \left| \frac{x_2 - x_1}{M_N} \right|$. Then, for a large N , we have $\frac{3\tau}{2} |\lambda_2 - \lambda_1| < 2\pi$. Therefore, $\tau \frac{|\lambda_2 - \lambda_1|}{4} < \frac{\tau |\lambda_2 - \lambda_1| + \tau \delta(N)}{2} < \pi - \tau \frac{|\lambda_2 - \lambda_1|}{4}$. Thus, we bound the integral as follows:

$$I_{2,i} \leq \frac{2\delta(N)n^{k\alpha}}{\left| \sin \frac{\tau |\lambda_2 - \lambda_1|}{4} \right|^{k\alpha}}$$

Since there is no v between $V_{j_i} + \delta(N)$ and $V_{j_{i+1}} - \delta(N)$ on which the denominators are vanishing, we get:

$$I_{3,i} \leq \frac{V_{j_{i+1}} - V_{j_i} - 2\delta(N)}{A \times B},$$

where

$$A = \inf \left(\left| \sin \frac{\tau \lambda_1 - \tau V_{j_i} - \tau \delta(N)}{2} \right|^{k\alpha}, \left| \sin \frac{\tau \lambda_1 - \tau V_{j_{i+1}} + \tau \delta(N)}{2} \right|^{k\alpha} \right)$$

$$B = \inf \left(\left| \sin \frac{\tau \lambda_2 - \tau V_{j_i} - \tau \delta(N)}{2} \right|^{k\alpha}, \left| \sin \frac{\tau \lambda_2 - \tau V_{j_{i+1}} + \tau \delta(N)}{2} \right|^{k\alpha} \right)$$

It follows from the hypothesis on $\delta(N)$ that

$$\frac{\tau \delta(N)}{2} < \frac{\tau |\lambda_1 - \lambda_2|}{2} - \frac{\tau |\delta(N)|}{2} < \frac{\tau |\lambda_1 - \lambda_2 - \delta(N)|}{2} < \frac{\tau |\lambda_1 - \lambda_2|}{2} + \frac{\tau |\delta(N)|}{2} < \pi - \frac{\tau \delta(N)}{2}.$$

Thus, by using the definition of V_{j_i} , we obtain

$$\frac{1}{\left| \sin \frac{\tau\lambda_2 - \tau V_{j_i} - \tau\delta(N)}{2} \right|^{k\alpha}} < \frac{1}{\left| \sin \frac{\tau\delta(N)}{2} \right|^{k\alpha}}.$$

We use the same way for bounding the other terms

$$I_{3,i} \leq \frac{V_{j_{i+1}} - V_{j_i} - 2\delta(N)}{\left| \sin \frac{\tau\delta(N)}{2} \right|^{2k\alpha}}.$$

and we get : $V_{j_{q+q'}} + \delta(N)$ Similarly, since there is no v between Ω and Ω which the denominators are vanishing, we can show that:

$$I_4 \leq \frac{V_{j_{i+1}} - V_{j_i} - 2\delta(N)}{E \times F}, \text{ where}$$

$$E = \inf \left(\left| \sin \frac{\tau\lambda_1 - \tau\Omega}{2} \right|^{k\alpha}, \left| \sin \frac{\tau\lambda_1 - \tau V_{j_{q+q'}} - \tau\delta(N)}{2} \right|^{k\alpha} \right)$$

$$F = \inf \left(\left| \sin \frac{\tau\lambda_2 - \tau\Omega}{2} \right|^{k\alpha}, \left| \sin \frac{\tau\lambda_2 - \tau V_{j_{q+q'}} - \tau\delta(N)}{2} \right|^{k\alpha} \right)$$

Since $\delta(N)$ converges to zero, for a large N , we have $\delta(N) < \frac{2}{\tau} \left(\pi - \frac{\tau}{2} |\lambda_1 - \Omega| \right)$, and

$$I_4 \leq \frac{\Omega - V_{j_{q+q'}} - \delta(N)}{\left| \sin \frac{\tau\delta(N)}{2} \right|^{2k\alpha}}.$$

$\delta(N) < \frac{2}{\tau} \left(\pi - \frac{\tau}{2} |\lambda_1 - \lambda_2| \right)$. It follows that:

We recapitulate, from the

previous increases, we obtain

$$\int_{-\Omega}^{\Omega} \left| \frac{\sin \left[\frac{n}{2} (\tau\lambda_1 - \tau v) \right]}{\sin \left[\frac{1}{2} (\tau\lambda_1 - \tau v) \right]} \right|^{k\alpha} \left| \frac{\sin \left[\frac{n}{2} (\tau\lambda_2 - \tau v) \right]}{\sin \left[\frac{1}{2} (\tau\lambda_2 - \tau v) \right]} \right|^{k\alpha} dv \leq \frac{\Omega + V_{j_1} - \delta(N)}{\left| \sin \frac{\tau\delta(N)}{2} \right|^{2k\alpha}} + \sum_{i=1}^{q+q'} \frac{2n^{k\alpha} \delta(N)}{\left| \sin \frac{\tau |\lambda_2 - \lambda_1|}{4} \right|^{k\alpha}}$$

$$+ \sum_{i=1}^{q+q'-1} \frac{V_{j_{i+1}} - V_{j_i} - \delta(N)}{\left| \sin \frac{\tau\delta(N)}{2} \right|^{2k\alpha}} + \frac{\Omega - V_{j_{q+q'}} - \delta(N)}{\left| \sin \frac{\tau\delta(N)}{2} \right|^{2k\alpha}}$$

After simplification, we have

$$\int_{-\Omega}^{\Omega} \left| \frac{\sin \left[\frac{n}{2} (\tau \lambda_1 - \tau v) \right]}{\sin \left[\frac{1}{2} (\tau \lambda_1 - \tau v) \right]} \right|^{k\alpha} \left| \frac{\sin \left[\frac{n}{2} (\tau \lambda_2 - \tau v) \right]}{\sin \left[\frac{1}{2} (\tau \lambda_2 - \tau v) \right]} \right|^{k\alpha} dv \leq \frac{2\Omega - 2(q+q'+1)\delta(N)}{\left| \sin \frac{\tau\delta(N)}{2} \right|^{2k\alpha}} + \frac{2n^{k\alpha} \delta(N)(q+q')}{\left| \sin \frac{\tau |\lambda_2 - \lambda_1|}{4} \right|^{k\alpha}}.$$

By bounding the first term on the right hand side of the last inequality and using the following

inequality $\left| \sin \frac{x}{2} \right| \geq \frac{|x|}{\pi}$, we get

$$\int_{-\Omega}^{\Omega} \left| \frac{\sin \left[\frac{n}{2} (\tau \lambda_1 - \tau v) \right]}{\sin \left[\frac{1}{2} (\tau \lambda_1 - \tau v) \right]} \right|^{k\alpha} \left| \frac{\sin \left[\frac{n}{2} (\tau \lambda_2 - \tau v) \right]}{\sin \left[\frac{1}{2} (\tau \lambda_2 - \tau v) \right]} \right|^{k\alpha} dv \leq \frac{2\Omega \pi^{2k\alpha}}{(\tau\delta(N))^{2k\alpha}} + \frac{2n^{k\alpha} \delta(N)(q+q')(2\pi)^{k\alpha}}{\left(\frac{\tau |x_2 - x_1|}{M_N} \right)^{k\alpha}}.$$

It follows from the lemma 2.1

$$\frac{1}{B'_{\alpha,N}} \int_{-\Omega}^{\Omega} \left| \frac{\sin \left[\frac{n}{2} (\tau \lambda_1 - \tau v) \right]}{\sin \left[\frac{1}{2} (\tau \lambda_1 - \tau v) \right]} \right|^{k\alpha} \left| \frac{\sin \left[\frac{n}{2} (\tau \lambda_2 - \tau v) \right]}{\sin \left[\frac{1}{2} (\tau \lambda_2 - \tau v) \right]} \right|^{k\alpha} dv \leq \frac{1}{\pi} \left(\frac{\pi}{2} \right)^{2k\alpha} \left(\frac{2\Omega \pi^{2k\alpha}}{n^{2k\alpha-1} (\tau\delta(N))^{2k\alpha}} + \frac{2\delta(N)(q+q')(2\pi)^{k\alpha}}{n^{k\alpha-1} \left(\frac{\tau\sigma_N}{M_N} \right)^{k\alpha}} \right). \quad (19)$$

In order to obtain the convergence of the last expression to zero, we choose $\delta(N) = n^{-\beta}$, $\beta > 0$, such as

$$\lim_{n \rightarrow \infty} \frac{n^{2k\alpha\beta}}{n^{2k\alpha-1}} = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{1}{n^{\beta+k\alpha-1} \left(\frac{\sigma_N}{M_N} \right)^{k\alpha}} = 0. \quad (20)$$

Thus, from (18) Q_N converges to zero. On the other hand,

$$C(\lambda) \leq F_{p,\alpha}^{-2} C_{\alpha}^{-\frac{2p}{\alpha}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |K - K'| e^{|K-K'|-K'} \frac{du_1 du_2}{|u_1 u_2|^{1+p}},$$

where $|K - K'| - K' \leq -C_{\alpha} \sum_{k'=1}^2 |u_{k'}|^{\alpha} |\psi_N(\lambda_{k'}) - \tau Q_N(\lambda_1; \lambda_2)|$.

We denote by : $\delta_{(N,k')} = \psi_N(\lambda_{k'}) - \tau Q_N(\lambda_1; \lambda_2)$. It follows from (18) and (20) that $\delta_{(N,k')}$ converges to $\phi(\lambda)$. Hence,

$$C(\lambda) \leq F_{p,\alpha}^{-2} C_\alpha^{-\frac{2p}{\alpha}} 2\tau C_\alpha Q_N(\lambda_1; \lambda_2) 4 \prod_{k'=1}^2 \int_0^\infty \exp[-C_\alpha (u_{k'})^\alpha |\delta_{(N,k')}|] \frac{du_{k'}}{(u_{k'})^{1+p-\frac{\alpha}{2}}}.$$

Putting $u_{k'} (\delta_{(N,k')})^{\frac{1}{\alpha}} = v$, we obtain

$$C(\lambda) \leq F_{p,\alpha}^{-2} C_\alpha^{-\frac{2p}{\alpha}} 2\tau C_\alpha \frac{Q_N(\lambda_1; \lambda_2)}{[\delta_{(N,1)} \delta_{(N,2)}]^{\frac{1-p}{\alpha}}} \left(\int_{-\infty}^\infty \frac{e^{-C_\alpha |v|^\alpha}}{|v|^{1+p-\frac{\alpha}{2}}} dv \right)^2. \quad (20)$$

Since $\phi(\lambda) > 0$, $C(\lambda)$ converges uniformly in $x_1, x_1' \in [-1, 1]$ to zero. From (20), we obtain

$$J_2 = O \left(\frac{1}{n^{2k\alpha(1-\beta)-1}} + \frac{1}{n^{\beta+k\alpha-1} \left(\frac{\sigma_N}{M_N} \right)^{k\alpha}} \right). \quad \text{Thus, } \text{var}[f_N(\lambda)] \text{ converges to zero and then, } f_N(\lambda)$$

is an asymptotically unbiased and consistent estimator.

4. CONCLUSIONS

In this paper, we estimate the spectral density of mixed stable process with continuous time when the process is observed at discrete instants. The aliasing phenomenon is avoided by assuming that the spectral is a compact support. This work can be applied in various fields. For example:

- The study of soil cracking where the observed signal is the resistance of the soil. This resistance can have random jumps that are due to the presence of some stones. Thus, the spectral measurement will be composed of two parts, one continuous and the other discrete corresponding to the resistance jumps encountered during the measurement.

- The growth of fruit on a tree can be considered as a continuous distribution, and when there is a fall of a fruit, the other fruits remaining on the tree absorb more energy and their growth will have a jump in value.

The perspective of this work is to optimize the smoothing parameters to have a better rate of convergence. For this purpose, the cross-validation method will be the most appropriate tool.

This work can also be completed by considering a more general case when we observe the process with random errors. In this case, we will use the deconvolution methods, which have proved their efficiency in the presence of random errors.

It would be interesting to give an estimator of the mode of the spectral density representing the frequency where the spectral density reaches the maximum of energy. For that, we must estimate the derivative of the spectral density function.

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